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# Chiral symmetry breaking from Ginsparg-Wilson fermions *

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We calculate the large-volume and small-mass dependences of the quark condensate in quenched QCD using Neuberger's operator. We find good agreement with the predictions of quenched chiral perturbation theory, enabling a determination of the chiral lagrangian parameter $\Sigma$, up to a multiplicative renormalization.

## 1. Introduction

Spontaneous chiral symmetry breaking (S$\chi$SB) is fundamental to our understanding of low energy hadronic phenomena and it is thus important to demonstrate quantitatively that it is a consequence of QCD. A natural candidate for such investigations is the numerical simulation of QCD on a spacetime lattice. S$\chi$SB, however, presents the lattice approach with a twofold challenge.

The first is that spontaneous symmetry breaking does not occur in a finite volume. In QCD, a possible signal of S$\chi$SB is the presence of a non-vanishing quark condensate defined as:

$$-\Sigma \equiv \langle\bar{q}q\rangle = \lim_{m\,\to\,0}\lim_{V\,\to\,\infty}\langle\bar{q}q\rangle_{m,V}\ , \tag{1}$$

where $\langle\bar{q}q\rangle_{m,V}$ is the condensate for finite volume $V$ and mass $m$. The double limit in Eq. (1) is rather challenging numerically! To get around this problem, we resort to a finite-size scaling analysis. This involves studying the scaling of the condensate with $V$ and $m$ as the limit of restoration of $\chi$S is approached ($m\to 0$, $V$ finite).

Such a study requires good control over the chiral properties of the theory, which is the second challenge. Indeed, at finite lattice spacing, "reasonable" discretizations of fermions either break continuum $\chi$S explicitly or lead to extraneous fermion species [2]. To minimize this problem, we resort to recently rediscovered [3,4] Ginsparg-Wilson (GW) fermions [5] which break continuum $\chi$S in a very mild and controlled fashion and actually have a slightly generalized $\chi$S even at finite lattice spacing [6].

## 2. Light quarks on a torus

In a large periodic box of volume $V=L^4$ such that $F_\pi L\gg 1$, for small quark masses and assuming the standard pattern of S$\chi$SB with $N_f\geq 2$, the QCD partition function is dominated by the nearly massless pions; the system can be described with the first few terms of a chiral lagrangian [7]. If, in addition, $m\to 0$ [5] so that $M_\pi L\simeq\frac{\sqrt{2m\Sigma}}{F_\pi}L\ll 1$, the global mode of the chiral lagrangian field $U\in SU(N_f)$ dominates the partition function, leading to a regime of restoration of $\chi$S [8].

In the quenched approximation to which we restrict here, topological zero modes of the Dirac operator induce $1/m$ singularities in $\langle\bar{q}q\rangle_{m,V}$ as $m\to 0$. To subtract these contributions, we work in sectors of fixed topological charge. Generalizing the line of argument given above, the partition function $Z_\nu$, in a sector of topological charge $\nu$, was recently evaluated [9] for the quenched case[6]. The quark condensate in sector $\nu$, proportional to the derivative of $\ln(Z_\nu)$ w.r.t. $m$, is then $-\Sigma_\nu\equiv\langle\bar{q}q\rangle_{m,V,\nu}$, such that [9]

$$\frac{\Sigma_\nu}{\Sigma}=z\left[I_\nu(z)K_\nu(z)+I_{\nu+1}(z)K_{\nu-1}(z)\right]+\frac{\nu}{z}, \tag{2}$$

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[5]We assume here for simplicity that the $N_f$ flavors all have mass $m$.

[6]The original unquenched treatment is given in [10].

where $z \equiv m\Sigma V$ and $I_\nu(z)$, $K_\nu(z)$ are the modified Bessel functions. As advertised, there is a divergence $\sim 1/m$ in sectors with topology. These terms, however, are independent of $\Sigma$.

Eq. (2) summarizes the scaling of the quark condensate with the volume and quark mass in the global mode regime, as a function of only one non-perturbative parameter: $\Sigma$. Thus, by fitting the dependence of the finite-volume condensate in quark mass and volume to Monte Carlo data, we can extract $\Sigma$ in a perfectly controlled manner.

## 3. Ginsparg-Wilson fermions

To perform the finite-volume scaling analysis outlined above, we need to be able to reach the chiral restoration regime without excessive fine-tuning and we need an index theorem to control the contribution of topological zero modes. Both these requirements are satisfied by GW fermions [11]. In particular, the leading cubic UV divergence of the condensate is known analytically for GW fermions and can thus be subtracted exactly. The resulting subtracted condensate, $\Sigma_\nu^{sub}$, however, is still divergent:

$$\Sigma_\nu^{sub}(a) = C_2 \frac{m}{a^2} + \cdots + \Sigma_\nu, \tag{3}$$

where $a$ is the lattice spacing. The coefficients of the divergences are not known a priori and have to be determined, preferably non-perturbatively. For the values of $m$ and $a$ considered below, however, only the quadratic divergence is important numerically, weaker divergences being suppressed by higher powers of $m$. A final multiplicative renormalization is still required to eliminate a residual logarithmic UV divergence in $\Sigma_\nu$.

In the present work, we use Neuberger's implementation of GW fermions encoded in the Dirac operator [12,4]:

$$aD_N = (1+s)\left[1 - A/\sqrt{A^\dagger A}\right] , \tag{4}$$

with $A = 1 + s - aD_W$ where $D_W$ is the standard Wilson-Dirac operator. The parameter $s$ must satisfy $|s| < 1$.

## 4. Numerical results

We work in the quenched approximation on hypercubic lattices with periodic boundary conditions for gauge and fermion fields. We choose $\beta = 5.85$, which corresponds to $a^{-1} \simeq 1.5$ GeV [14], and use standard methods to obtain decorrelated gauge-field configurations.

To evaluate $1/\sqrt{A^\dagger A}$ in Eq. (4), we use a Chebyshev approximation, $P_{n,\epsilon}(A^\dagger A)$, where $P_{n,\epsilon}$ is a polynomial of degree $n$, which gives an exponentially converging approximation to $1/\sqrt{x}$ for $x \in [\epsilon, 1]$ [13]. The cost of a multiplication by $D_N$ is linear in $n$ and becomes rapidly high. To reduce $n$ substantially, we perform the improvements described in [1]. We take $s = 0.6$, a value at which Neuberger's operator is nearly optimally local for $\beta = 5.85$ [13].

To determine whether a gauge configuration belongs to the $\nu = 0$ or $\pm 1$ sectors, we compute the few lowest eigenvalues of $D_N^\dagger D_N$ by minimizing the relevant Ritz functional [16]. As pointed out in [15], it is advantageous for this computation, as well as for the inversion of $(D_N^\dagger + m)(D_N + m)$, to stay in a given chiral subspace. Having determined the topological charge of a configuration, we then obtain the condensate of Eq. (3) in three volumes ($8^4$, $10^4$ and $12^4$) by computing

$$\Sigma_\nu^{sub} = \frac{1}{V}\left\langle \mathrm{Tr}'\left\{\frac{1}{D_N + m} + \mathrm{h.}c. - \frac{a}{1+s}\right\}\right\rangle_\nu , \tag{5}$$

where the trace is taken in the chiral sector opposite to that with the zero modes [15] and the gauge average is performed in a sector of fixed topology $\nu$. With this definition, terms $\sim 1/m$ in Eq. (2) are absent[7]. Three gaussian sources and a multimass solver [17] were used to compute the trace in Eq. (5) for seven values of $m$.

We show in Fig. 1 our results for $a^3\Sigma_{\nu=\pm 1}^{sub}/am$ as a function of bare quark mass. We have 15, 10 and 7 gauge configurations on our $8^4$, $10^4$ and $12^4$ lattices, respectively[8]. The solid lines are a fit of the data to Eqs. (3) and (2) for all volumes and

[7] Though not shown explicitly in Eq. (5), we correctly account for the real eigenvalues of $D_N$ at $2(1+s)/a$.

[8] For the larger volumes, $\nu = 0$ configurations are rare while the calculation of $\Sigma_{\nu=0}^{sub}$ requires large statistics [1]. $|\nu| > 1$ configurations, on the other hand, are rare in the smaller volumes.

masses. This fit has only two parameters, namely $\Sigma$ and the coefficient of the quadratic divergence. We find $a^3\Sigma = 0.0032(4)$ and $C_2 = -0.914(8)$.

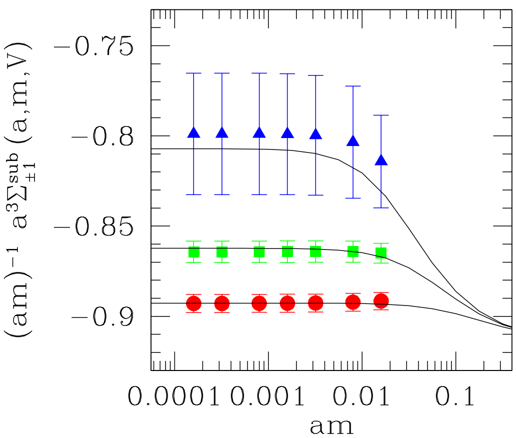


Figure 1. *Mass dependence of the condensate for the* $8^4$ *(circles),* $10^4$ *(squares) and* $12^4$ *(triangles) lattices. The curves result from a fit to Eqs. (3) and (2).*

Clearly, the formulae derived in quenched $\chi$PT give a very good description of the numerical data. The value of $\Sigma$ that we extract is, in physical units, $\Sigma(\mu \sim 1.5\ \text{GeV}) = (221^{+8}_{-9}\ \text{MeV})^3$, up to a multiplicative renormalization constant, which has not been computed yet for Neuberger's operator. The quoted error on $\Sigma$ is purely statistical and the statistics are rather small. Quenching and discretization errors, for instance, as well as possible contributions from higher orders in $\chi$PT are not accounted for. Nevertheless, the value obtained and the agreement with q$\chi$PT support the standard scenario of S$\chi$SB.

We further consider the mean value of the lowest non-zero eigenvalue of $\sqrt{D_N^\dagger D_N}$ in different topological sectors. In Random Matrix Theory the distributions of these eigenvalues are given solely in terms of $\Sigma$ [18]. Our determination of $\Sigma$ therefore yields predictions for the mean values. These can then be compared to the average values obtained in simulation. With our $8^4$ results, we find agreement within roughly one standard deviation for $|\nu| = 1$ (29 configurations) and two for $\nu = 0$ (41 configurations) [1].

*Note: Related work with Neuberger's operator can be found in [15,19,20].*

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